Benha University Faculty of Engineering- Shoubra Energy and Sustainable Energy Department		Final Term Exam Date: Course: Mathematics III Duration: **3** hours

- Answer **All** questions
- The Exam Consists of One page
- No. of questions: 4, (10) marks for each
- Total Mark: 40 Marks

Answer the following questions

(1) (a) Find the parametric equation for the curve $r = 4\cot\theta\csc\theta$ hence find the area bounded by the curve in the region $\boldsymbol{x} = 0$ to $\boldsymbol{x} = 1$. 5

(b) Find the volume of the solid generated by revolving, the region between the curve: x = cos t, y = sin t , t in [0, π/2], about x- axis. 5

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(2) (a) Find $\emptyset(x, y, z)$ which satisfy $\vec{\nabla}\emptyset = 2xyz^3\vec{\imath} + x^2xz^3\vec{\jmath} + 3x^2yz^2\vec{k}$, hence find $\vec{\nabla}.\vec{\nabla}\emptyset$ 5

(b) Verify **Green's theorem** for the integral $\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)\,dy$

Where C the region enclosed by $x + y = 1\; x = 0,\; y = 0$. 5

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(3) (a) Find the Fourier series of the function $f(x) = |x|$ which satisfy $f(x) = f(x + 2\pi)$. 5

(b) Given $u = e^x \sin y$ find the function $\boldsymbol{v}$ such that $f(z) = u + iv$ is analytic function. 5

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(4) (a) prove that the functions $f(z) = \sin z$ satisfy Cauchy Riemann equations, hence show that $\overline{\sin z} = \sin \bar{z}$ 5

(b) By using Cauchy integral Formula evaluate the following integrals 5

(i) $\oint_C \frac{z\,dz}{(9 - z^2)(z + i)}$ on $|z| = 2$

(ii) $\oint_C \frac{z^4 - 3z^2 + 6}{(z + i)^3}\,dz$ on $|z| = 4$

Good Luck

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Answer of question (1)

(a) Since $r = 4\cot\theta\csc\theta$

Then

$$x = r\cos\theta = 4\cot\theta\csc\theta\cos\theta = 4\cot^2\theta$$
$$y = r\sin\theta = 4\cot\theta\csc\theta\sin\theta = 4\cot\theta$$

This is the parametric equation

If we put $t = \cot\theta$ The parametric equation simplified to

$$x = 4t^2$$
$$y = 4t$$

Which represent a parabola symmetric about ***x***-axis, When ***x***=0 , ***t***=0 when ***x***=1 ***t*** =±1/2

Change the limit of integration of x to the value of t we have

Required area is $A = \int_0^1 y dx = 2\int_0^{1/2} 4t(8t)dtdx = 64\left[\frac{t^3}{3}\right]_0^{1/2} = \frac{64}{3}\left[\left(\frac{1}{2}\right)^3 - 0\right] = \frac{8}{3}$

Another solution

$$A = 2\int y dx = 2\int 4t(8t)dtdx = 64\frac{t^3}{3}$$

Transfer the result to ***x*** and substitute by the value of ***x*** in the result as following

$$64\frac{t^3}{3} = \frac{64}{3}\left[\left(\frac{x}{4}\right)^{3/2}\right]_0^1 = \frac{64}{3}\left[\left(\frac{1}{4}\right)^{3/2} - 0\right] = \frac{8}{3}$$

(b) $V = \int_{t=0}^{t=\pi/2} \pi y^2 dx = -\int_{t=0}^{t=\pi/2} \pi\sin^2 x\sin x dx = -\pi\int_{t=0}^{t=\pi/2} (1-\cos^2 x)\sin x dx$

$$= -\pi\int_{t=0}^{t=\pi/2} (\sin x - \cos^2 x\sin x)dx = -\pi\left[(\cos x - \frac{1}{3}\cos^3 x)\right]_0^{\pi/2}$$

$$= -\pi(\cos\frac{\pi}{2} - \frac{1}{3}\cos^3\frac{\pi}{2}) + \pi(\cos 0 - \frac{1}{3}\cos^3 0) = \pi(1-\frac{1}{3}) = \frac{2}{3}\pi$$

Answer of question (2)

$\vec{\nabla}\emptyset = 2xyz^3\vec{\imath} + x^2yz^3\vec{\jmath} + 3x^2yz^2\vec{k}$, then

$\frac{\partial\emptyset}{\partial x}\vec{\imath} + \frac{\partial\emptyset}{\partial y}\vec{\jmath} + \frac{\partial\emptyset}{\partial z}\vec{k} = 2xyz^3\vec{\imath} + x^2yz^3\vec{\jmath} + 3x^2yz^2\vec{k}$,

$\frac{\partial\emptyset}{\partial x} = 2xyz^3$, $\frac{\partial\emptyset}{\partial y} = x^2xz^3 and$ $\frac{\partial\emptyset}{\partial z} = 3x^2yz^2$,

$by\ integration\ \ \emptyset = x^2yz^3$ and $\vec{\nabla}.\vec{\nabla}\emptyset = 2yz^3 + x^2z^3 + 6x^2yz$

(c) $$I = \oint_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy = \int_{\substack{(0,0)\\ on\,y=0}}^{(1,0)} (3x^2-8y^2)dx + (4y-6xy)\,dy$$

$$+\int_{\substack{(1,0)\\ on\,y=1-x}}^{(0,1)} (3x^2-8y^2)dx + (4y-6xy)\,dy + \int_{\substack{(0,1)\\ on\,x=0}}^{(0,0)} (3x^2-8y^2)dx + (4y-6xy)\,dy$$

$$= \int_0^1 (3x^2)dx + \int_1^0 \left(3x^2 - 8(1-x)^2\right)dx + \left(4(1-x) - 6x(1-x)\right)(-dx) + \int_1^0 4y\,dy = \frac{5}{3}$$

<u>**Calculation of the double integral:**</u>

Green's theorem states that $\boxed{\oint_C M(x,y)dx + N(x,y)dy = \iint_S \left(\frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y}\right)dxdy}$

$$\therefore \oint_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$$

$$= \iint_R \left[\frac{\partial}{\partial x}(4y-6xy) - \frac{\partial}{\partial y}(3x^2-8y^2)\right]dxdy = \iint_R \left[-6y+16y\right]dxdy$$

$$= \iint_R \left[10y\right]dxdy = \int_0^1\int_0^{1-x} (10y)dxdy = \int_0^1 5y^2\Big|_0^{1-x} dx = \int_0^1 \left[5(1-x)^2\right]dx = \left[-\frac{5}{3}(1-x)^3\right]_0^1 = \frac{5}{3}$$

Then the line integral verify **Green's theorem**

Answer of question (3)

(a) $f(x)=|x|, \qquad -\pi<x<\pi$

Solution:

$\boldsymbol{f(x)}$ is an even function then $b_n=0,\ n=1,2,3,...$

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx=\frac{1}{2\pi}\int_{-\pi}^{\pi} xdx=\frac{1}{\pi}\int_{0}^{\pi} xdx=\frac{1}{\pi}\left[\frac{x^2}{2}\Big|_0^{\pi}\right]=\frac{1}{\pi}\left[\frac{\pi^2}{2}\right]=\frac{\pi}{2}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx=\frac{2}{\pi}\int_{0}^{\pi} x\cos nx\,dx$$

$$=\frac{2}{\pi}\left[\frac{x}{n}\sin nx+\frac{1}{n^2}\cos nx\right]_0^{\pi}=\frac{2}{\pi}\left[\frac{1}{n^2}\cos n\pi-\frac{1}{n^2}\right]=\frac{2}{\pi n^2}\left[(-1)^n-1\right]$$

$$a_{2n}=0, a_{2n-1}=\frac{-4}{\pi(2n-1)^2}$$

$$f(x)=a_0+\sum_{n=0}^{\infty} a_n\cos nx \qquad \boxed{|x|=\frac{\pi}{2}+\sum_{n=0}^{\infty}\frac{-4}{\pi(2n-1)^2}\cos(2n-1)x}$$

(b) So that $\boldsymbol{u}$ is harmonic now we obtain the function $\boldsymbol{v}$ such that $\boldsymbol{u}$ and $\boldsymbol{v}$ satisfies the equations $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ Integrate the **first equation with respect to y**

$v=\int\frac{\partial u}{\partial x}\,dy+f(x)=\int e^x\sin y\,dy+f(x)=-e^x\cos y+f(x)$ where $f(x)$ is the integration constant and to determine $f(x)$ we use second equation as follows

$$\frac{\partial v}{\partial x}=-e^x\cos y+f'(x)=-\frac{\partial u}{\partial y}=-e^x\cos y\ \ \therefore f'=0\Rightarrow f(x)=C\ \text{(pure arbitrary constant)}$$

Hence $v=-e^x\cos y$ and $f(z)=u+iv=e^x\sin y-ie^x\cos y=-ie^x(\cos y+i\sin y)=-ie^z$

where $\boldsymbol{C}$ is an arbitrary constant.

Answer question (5)

(a) prove that the functions $f(z)=\sin z$ satisfy Cauchy Riemann equations,

hence show that $\overline{\sin z}=\sin\bar{z}$

(b) By using Cauchy integral Formula evaluate the following integrals

$\oint_C \frac{zdz}{(9-z^2)(z+i)}$ (b) $\oint_C \frac{z^4-3z^2+6}{(z+i)^3}\,dz$

(a) $f(z)=\sin z=\sin(x+iy)=\sin x\cosh y+i\cos x\sinh y$

$u=\sin x\cosh y \qquad v=\cos x\sinh y$

$\frac{\partial u}{\partial x}=\cos x\cosh y \qquad \frac{\partial v}{\partial x}=-\sin x\sinh y$

$\frac{\partial u}{\partial y}=\sin x\sinh y \qquad \frac{\partial v}{\partial y}=\cos x\cosh y$

$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

Then $f(z)=\sin z$ satisfy Cauchy Riemann equations,

$$\begin{aligned}\overline{\sin z}&=\overline{\sin(x+iy)}\\&=\overline{\sin x\,\cos(iy)+\cos x\,\sin(iy)}=\overline{\sin x\,\cosh y+i\cos x\,\sinh y}\\&=\sin x\,\cosh y-i\cos x\,\sinh y=\sin x\,\cos(iy)-\cos x\,\sin(iy)\\&=\sin(x-iy)=\sin\bar{z} \qquad \text{then } \overline{\sin z}=\sin\bar{z}\end{aligned}$$

(b) $\oint_C \frac{zdz}{(9-z^2)(z+i)}$

Since $\boldsymbol{z=-i}$ inside the integral curve $\boldsymbol{C}$ and $f(z)=\frac{z}{9-z^2}$ analytic inside and on the

circle then $\oint_C \frac{zdz}{(9-z^2)(z+i)}=2\pi i\left(\frac{z}{9-z^2}\right)_{at\ z=i}=2\pi i\left(\frac{-i}{9-(-i)^2}\right)=\frac{\pi}{5}$

Since $\boldsymbol{z=-i}$ inside the integral curve $\boldsymbol{C}$ then

$$\oint_C \frac{z^4-3z^2+6}{(z+i)^3}\,dz=\left[\pi i(z^4-3z^2+6)''\right]_{z=-i}=\pi i\left[12z^2-6\right]_{z=-i}=-18\pi i$$